Inverse Trigonometric Functions

Inverse trig functions are hard, in part, because they're not actually inverses. They're only partial inverses. Remember how, in order to have an inverse, a function must be 1-1? Well, take a look at the sine function – it's about as far away from 1-1 as you can get. If you draw a horizontal line through the graph of the sine function, it will hit the graph an infinite number of times. You might say the sine function is ∞-1, rather than 1-1. However, that doesn't have to stop us from finding a *partial* inverse of the sine function. You've done this before, finding partial inverses of quadratic functions and absolute value functions. We just have to restrict the domain.

The easiest way to go about figuring this out is by looking at a graph. Below is a graph of the sine function.

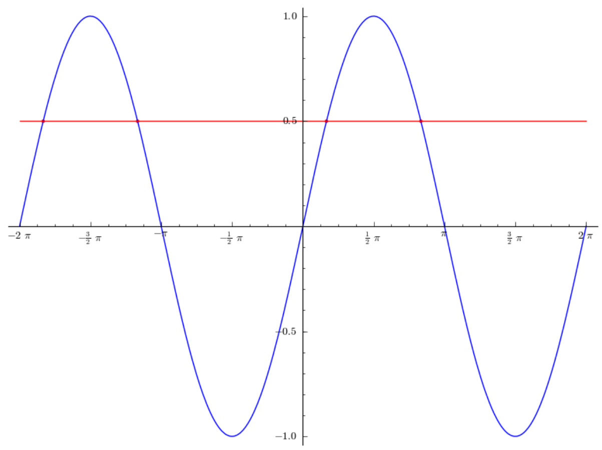


FIGURE 1.

We need to restrict this so that any horizontal line only hits the graph once. An obvious choice of restricted domain is to restrict to only the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

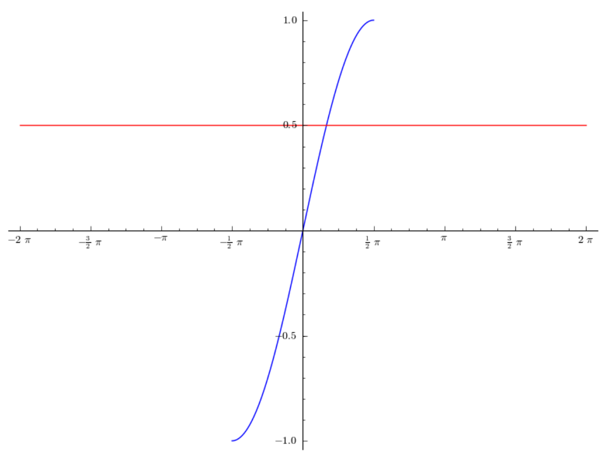


FIGURE 2.

This restricted sine function is what we can find an inverse of. Finding the domain, range, and graph of the inverse sine function is just like any inverse function. The domain of the (restricted) sine function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, so the range of its inverse is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The range of the (restricted) sine function is $[-1, 1]$, so the domain of its inverse is $[-1, 1]$. The graph of the inverse is just the graph of the original flipped over the line $y = x$. Try to draw it yourself.

There are two notations for inverse trig functions. One is the typical "$-1''$" we use for all inverse functions, i.e., $\sin^{-1}(x)$. The other is "arc" notation. Another name for the inverse sine function is the "arcsine" function, and we write $\arcsin(x)$; $\arcsin(x)$ and $\sin^{-1}(x)$ mean exactly the same thing. We'll use them both so you get used to both notations.

Where things may get more tricky is in the interpretation of inverse trig functions. For example, what is $\sin^{-1}(-\frac{1}{2})$? We can figure it out by using the relationship between a function an its inverse. If $\sin^{-1}(-\frac{1}{2}) = y$, that means that $\sin(y) = -\frac{1}{2}$, by definition of inverse functions. So what are some values of y where $\sin(y) = -\frac{1}{2}$? You've done enough solving of trig equations that this should be easy – y could be $\frac{7\pi}{6}$, or $\frac{11\pi}{6}$, or $-\frac{\pi}{6}$, and so on. We need to pick just one of these values, however, because we're trying to find the output of the inverse sine function when you plug in $-\frac{1}{2}$, and a function can only have one output for each input. Which one do we choose? Look back at the definition of the inverse sine function – its domain is $[-1, 1]$ and its range, it's possible outputs, is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. There's only one possible output in that range, and that is $-\frac{\pi}{6}$. Therefore, $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$.

It may help to say out in words what the inverse sine function does:

"The inverse sine of x, $\sin^{-1}(x)$, is an angle θ such that $\sin(\theta) = x$ and θ is in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$."

That's a complicated statement, but it's as simple as it gets. The first part ("such that $\sin(\theta) = x$") is the definition of an inverse function, and the second part ("θ is in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$") comes from the fact that we restricted the domain of the sine function to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The other trig functions also have to have their domains restricted, since none of them are 1-1, but the choice of domain differs depending on the function. For the cosine function, we restrict its domain to $[0, \pi]$. For the tangent function, we restrict its domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (same as sine). The other three trig functions are even weirder, so we'll just focus on these three for this class.

Evaluate the following quantities.

(1) $\sin^{-1}(-1)$

(2) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

(3) $\arctan(\sqrt{3})$

(4) $\cos^{-1}(\frac{1}{2})$

(5) $\tan^{-1}(-\frac{\sqrt{3}}{3})$

Because of the restricted domain action going on, inverse trig functions don't always act exactly like inverse functions are supposed to. Everyone knows that $f^{-1}(f(x)) = x$, for example. And much of the time, this is true. For example, $\sin^{-1}(\sin(\frac{\pi}{4})) = \frac{\pi}{4}$. However, it's not always true. For example, $\sin^{-1}(\sin(\frac{3\pi}{4})) = \frac{\pi}{4}$. What's going on?

The issue lies in the fact, again, that inverse trig functions are only partial inverses. Take a look at the expression $\sin^{-1}(\sin(\frac{3\pi}{4}))$ in more detail. What is the sine of $\frac{3\pi}{4}$? By the unit circle, it must be $\frac{\sqrt{2}}{2}$. What is the inverse sine of $\frac{\sqrt{2}}{2}$? That is an angle θ such that $\sin(\theta) = \frac{\sqrt{2}}{2}$ *and* θ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Now, $\frac{3\pi}{4}$ is an angle such that $\sin(\theta) = \frac{\sqrt{2}}{2}$, but it does not lie in the proper interval. Only one angle does, and that is $\frac{\pi}{4}$. The lesson here is that you must be careful with inverse trig functions, since they're not true inverses.

Evaluate the following. Hint: for the last two, drawing a right triangle may help.

(6) $\cos(\arcsin(-\frac{\sqrt{2}}{2}))$

(7) $\tan^{-1}(\cos(\pi))$

(8) $\arccos(\cos(\frac{9\pi}{4}))$

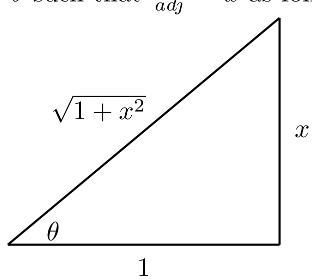
(9) $\sin(\arcsin(.32))$

(10) $\sin^{-1}(\sin(\frac{7\pi}{6}))$

(11) $\cos(\arcsin(\frac{3}{4}))$

(12) $\tan(\cos^{-1}(\frac{5}{7}))$

Using all the facts at our disposal, we can even simplify variable expressions like $\sin(\arctan(x))$; that is, we can come up with a formula for such a thing. We know that $\arctan(x) = \theta$ with $\tan(\theta) = x$ and $\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We can set up a right triangle with angle θ such that $\frac{opp}{adj} = x$ as follows:



With our triangle set up, it is clear to see that $\sin(\arctan(x)) = \sin(\theta) = \dfrac{x}{\sqrt{1+x^2}}$.

Now you try. Find formulas for the following.

(13) $\tan(\arccos(x))$

(14) $\sec(\arcsin(x))$